Determination of the complete set of shaking force and shaking moment balanced planar four-bar linkages

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Abstract

A mechanism is said to be force balanced if, for any arbitrary motion, it does not apply reaction forces on the base. Moreover, if it does not apply torques on the base, the mechanism is said to be moment balanced or dynamically balanced. In this paper, a new method to determine the complete set of force and moment balanced planar four-bar linkages is presented. Using complex variables to model the kinematics of the linkage, the force and moment balancing constraints are written as algebraic equations over complex variables and joint angular velocities. Using polynomial division, necessary and sufficient conditions for the balancing of planar four-bar linkages are derived.

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Key words: shaking force balancing, shaking moment balancing, dynamic balancing, planar four-bar mechanisms.

1 Introduction

A mechanism is said to be force balanced if, for any motion of the mechanism, no reaction forces other than gravity are transmitted to its base. A mechanism is said to be force and moment balanced if no reaction force or moment is transmitted to the base, at any time, for any arbitrary motion of the mechanism [15]. The latter property is also referred to as dynamic balancing. Force and moment balanced mechanisms are highly desirable for many engineering applications in order to reduce fatigue, vibrations and wear. Dynamic balancing can also be used in more advanced applications such as, for instance, the design of compensation mechanisms for telescopes. Additionally, dynamic balancing is very attractive for space applications since the reaction forces and torques induced at the base of space manipulators or mechanisms are one of the reasons why the latter are constrained to move very slowly [14].

The compensation of shaking forces and shaking moments — the reaction forces and moments transmitted to the base of a mechanism — has been a subject of research for several decades (see for instance [7,11,12]). The problem of force balancing was first addressed by Berkof and Lowen [3], who provided conditions for force balancing in terms of the design parameters when the geometric parameters are sufficiently generic. A non-generic solution was then

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found in [8]. Indeed, in the latter reference, special cases of force balanced four-bar linkages that do not satisfy the general conditions derived in [3] were revealed. The problem of dynamic balancing was addressed in [4, 5] for the generic case. It was shown that additional counterrotations are required in order to balance the shaking forces and moments of a planar four-bar linkage. Based on this concept, several balancing techniques were developed in the literature (see for instance [1, 2]). However, in [13] and [9], special cases that do not require external counterrotations were first revealed, but no complete characterisation of all cases was given. Furthermore, the mathematical techniques used to obtain the dynamically balanced mechanisms cannot guarantee that all possible solutions are found.

The aim of this paper is to derive all possible sets of design parameters for which a planar four-bar linkage is dynamically balanced without counterrotations. The paper is organised as follows. In section 2 we derive a system of algebraic equations in terms of the design parameters and the joint angles in the configuration space — modelled by complex variables $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$ — and eliminate variable $\mathbf{z}_3$ immediately. In section 3, the remaining variables $\mathbf{z}_1, \mathbf{z}_2$ are eliminated and conditions on the design parameters are obtained. The method is applied for the generic case as well as for all possible specific (reducible) cases, namely: the parallelogram, the deltoid and the rhomboid. This leads to a complete description of all possible dynamically balanced planar four-bar linkages. The technique used ensures that the description is exhaustive.

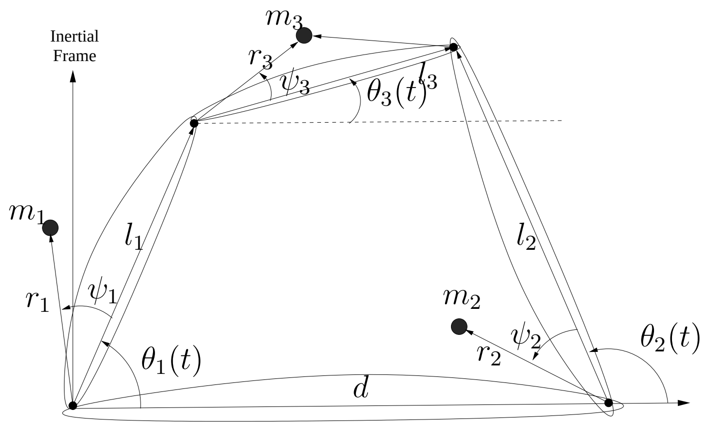


Fig. 1. Four-bar linkage.

2 Problem formulation

2.1 Representation of planar four-bar linkages

A planar four-bar linkage is shown in Fig. 1. It consists of four links: the base of length d which is fixed, and three moveable links of length l_1, l_2, l_3 respectively. We assume that all link lengths are strictly positive. Since the base is fixed, the mass properties of the base has no influence on the equations and will therefore be ignored. Each of the three moveable links has a mass m_i, a centre of mass whose position is defined by r_i and ψ_i and the axial moment of inertia I_i. The design of planar four-bar linkages consists in choosing the 16 design parameters shown in Table 1.

The links are connected by revolute joints rotating about axes pointing in a direction orthogonal to the plane of motion. The joint angles are specified using the time variables $\theta_1(t), \theta_2(t)$ and $\theta_3(t)$ as shown in Fig. 1. The kinematics of planar linkages can be conveniently represented in the complex plane, using complex numbers to describe the linkage's configuration (Fig. 2) and the position of the centre of mass (Fig. 3). Referring to Figs. 2 and 3, let $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$

Type		Parameters
Geometric	Length	l_1, l_2, l_3, d
Static	Mass	m_1, m_2, m_3
	Centre of mass	$r_1, \psi_1, r_2, \psi_2, r_3, \psi_3$
Dynamic	Inertia	I_1, I_2, I_3

Table 1

Design parameters for the planar four-bar linkages.

be time dependent unit complex numbers and $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ unit complex numbers depending on the design parameters (actually only on ψ_1, ψ_2, ψ_3). The orientation of $\mathbf{p}_i$ is specified relative to $\mathbf{z}_i$, i.e., it is attached to $\mathbf{z}_i$ and moves with it. If $\mathbf{p}_i$ coincides with $\mathbf{z}_i$, then $\mathbf{p}_i = 1$.

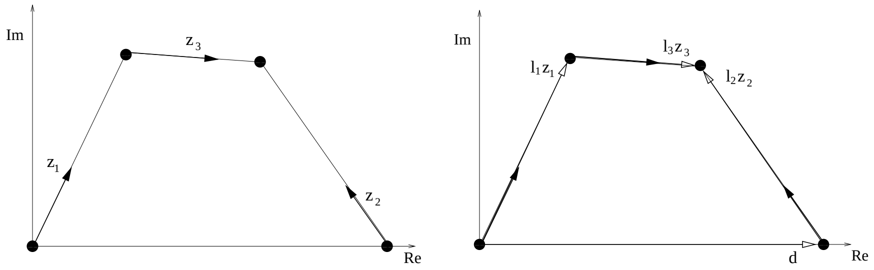


Fig. 2. Complex representation for the kinematics.

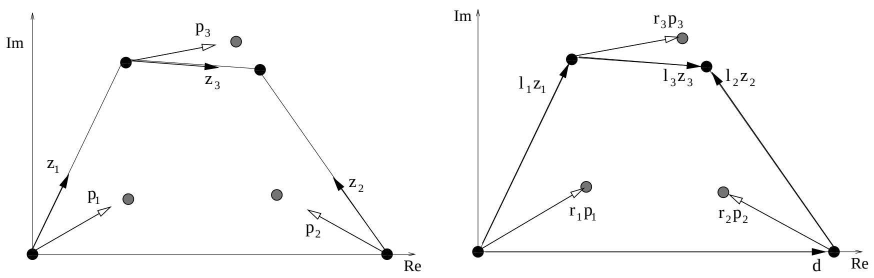


Fig. 3. Complex representation for the centre of masses.

2.2 Kinematic model

The dependency between the different joint angles is described by the following closure constraint:

$$\mathbf{z}_3 = G_1\mathbf{z}_1 + G_2\mathbf{z}_2 + G_3 \tag{1}$$

where $G_1, G_2, G_3 \in \mathbb{R}$ with $G_1 = \frac{-l_1}{l_3}, G_2 = \frac{l_2}{l_3}, G_3 = \frac{d}{l_3}$. Taking the time derivative of (1), we get a relationship between the joint angular velocities $\dot{\theta}_1, \dot{\theta}_2$ and $\dot{\theta}_3$, namely:

$$\mathbf{z}_3\dot{\theta}_3 = G_1\mathbf{z}_1\dot{\theta}_1 + G_2\mathbf{z}_2\dot{\theta}_2 \tag{2}$$

Since $\mathbf{z}_3$ is a unit complex number, $\mathbf{z}_3\overline{\mathbf{z}_3} = \mathbf{z}_3\mathbf{z}_3^{-1} = 1$ and therefore we obtain the following geometric constraint:

$$
\begin{aligned}
G &= (G_1\mathbf{z}_1 + G_2\mathbf{z}_2 + G_3)\left(G_1\mathbf{z}_1^{-1} + G_2\mathbf{z}_2^{-1} + G_3\right) - 1 \\
&= G_1G_2(\mathbf{z}_1^{-1}\mathbf{z}_2 + \mathbf{z}_1\mathbf{z}_2^{-1}) + G_1G_3(\mathbf{z}_1 + \mathbf{z}_1^{-1}) \\
&\qquad + G_2G_3(\mathbf{z}_2 + \mathbf{z}_2^{-1}) + (G_1^2 + G_2^2 + G_3^2 - 1) = 0
\end{aligned}
\tag{3}
$$

The time derivative of the geometric constraint (3) can be written as a linear combination of the joint angular velocities:

$$
i(K_1\dot{\theta}_1 + K_2\dot{\theta}_2) = 0 \tag{4}
$$

where

$$
K_1 = G_1G_2(\mathbf{z}_1\mathbf{z}_2^{-1} - \mathbf{z}_1^{-1}\mathbf{z}_2) + G_1G_3(\mathbf{z}_1 - \mathbf{z}_1^{-1}) \tag{5}
$$

$$
K_2 = G_1G_2(\mathbf{z}_1^{-1}\mathbf{z}_2 - \mathbf{z}_1\mathbf{z}_2^{-1}) + G_2G_3(\mathbf{z}_2 - \mathbf{z}_2^{-1}) \tag{6}
$$

It is noted that since K_1 and K_2 are purely imaginary, only one constraint equation is obtained, over the real set.

2.3 Position of the centre of mass

Let M be the total mass of the linkage ($M = m_1 + m_2 + m_3$). The position of the centre of mass of the linkage $\mathbf{rS}$ is:

$$
\mathbf{rS} = \frac{1}{M}\left(\mathbf{rS}_1 + \mathbf{rS}_2 + \mathbf{rS}_3\right) \tag{7}
$$

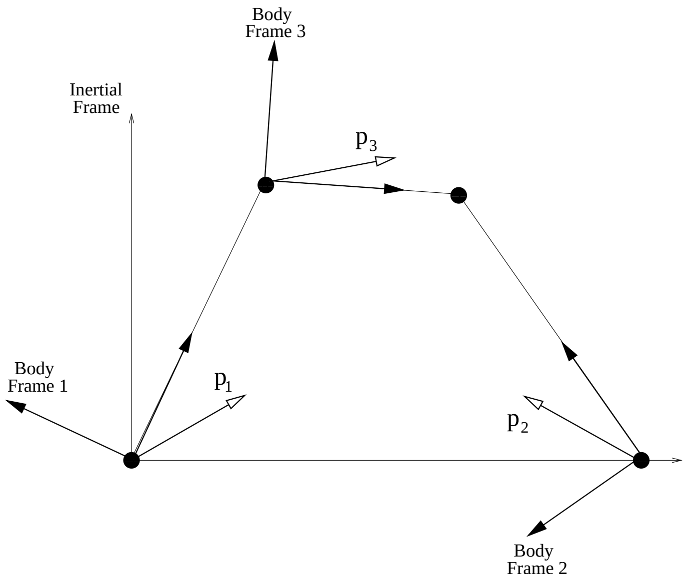


Fig. 4. Unit vectors representation

where $\mathbf{rS}_1$, $\mathbf{rS}_2$ and $\mathbf{rS}_3$ are the positions of the centre of mass of the three moving links expressed in the reference frame:

$$\begin{aligned} \mathbf{rS}_1 &= m_1 r_1 \mathbf{p}_1 \mathbf{z}_1 \\ \mathbf{rS}_2 &= m_2 \left(d + r_2 \mathbf{p}_2 \mathbf{z}_2 \right) \\ \mathbf{rS}_3 &= m_3 \left(l_1 \mathbf{z}_1 + r_3 \mathbf{p}_3 \mathbf{z}_3 \right) \end{aligned} \tag{8}$$

Substituting (1) in (7), variable $\mathbf{z}_3$ can be eliminated and the position of the centre of mass can be written in the following form:

$$\mathbf{rS} = \frac{1}{M} \left(\mathbf{F}_1 \mathbf{z}_1 + \mathbf{F}_2 \mathbf{z}_2 + \mathbf{F}_3 \right) \tag{9}$$

where $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3 \in \mathbb{C}$:

$$\begin{aligned} \mathbf{F}_1 &= m_1 r_1 \mathbf{p}_1 + m_3 l_1 + G_1 m_3 r_3 \mathbf{p}_3 \\ \mathbf{F}_2 &= m_2 r_2 \mathbf{p}_2 + G_2 m_3 r_3 \mathbf{p}_3 \\ \mathbf{F}_3 &= m_2 d + G_3 m_3 r_3 \mathbf{p}_3 . \end{aligned} \tag{10}$$

2.4 Angular momentum

Since the linkage is planar, the contribution of body i to the angular momentum is a scalar and can be given in the following form:

$$H_i = m_i \langle r_i, -i\dot{r_i}\rangle + I_i\dot{\theta}_i \tag{11}$$

where r_i and $\dot{r_i}$ are respectively the position and the velocity of the centre of mass of body i with respect to a given inertial frame, I_i denotes the axial moment of inertia of body i with respect to its centre of mass and $\langle *, *\rangle$ is the scalar product of planar vectors, i.e. $\langle u, v\rangle = \mathrm{Re}(u\overline{v}) = \frac{u\overline{v}+\overline{u}v}{2}$. The total angular momentum H of the system is given by the sum of the angular momentum of the links, (i.e. $H = H_1 + H_2 + H_3$).

The angular momentum of the first body with respect to the inertial frame is:

$$H_1 = \langle r_1\mathbf{p}_1\mathbf{z}_1, -im_1(ir_1\mathbf{p}_1\mathbf{z}_1\dot{\theta}_1)\rangle + I_1\dot{\theta}_1 = \langle r_1\mathbf{p}_1\mathbf{z}_1, m_1r_1\mathbf{p}_1\mathbf{z}_1\dot{\theta}_1\rangle + I_1\dot{\theta}_1 = J_1\dot{\theta}_1 \tag{12}$$

The contribution of the second body to the angular momentum is given by:

$$H_2 = \Big\langle (d + r_2\mathbf{p}_2\mathbf{z}_2), m_2r_2\mathbf{p}_2\mathbf{z}_2\dot{\theta}_2\Big\rangle + I_2\dot{\theta}_2 = \left[m_2dr_2\left(\frac{\mathbf{p}_2\mathbf{z}_2 + \mathbf{p}_2^{-1}\mathbf{z}_2^{-1}}{2}\right) + J_2\right]\dot{\theta}_2 \tag{13}$$

For the third body, we get

$$H_3 = \Big\langle (l_1\mathbf{z}_1 + r_3\mathbf{p}_3\mathbf{z}_3), m_3\left(l_1\mathbf{z}_1\dot{\theta}_1 + r_3\mathbf{p}_3\mathbf{z}_3\dot{\theta}_3\right)\Big\rangle + I_3\dot{\theta}_3 \tag{14}$$

Substituting (1) and (2) into (14), we can eliminate $\mathbf{z}_3\dot{\theta}_3$ and $\dot{\theta}_3$ and obtain an expression in terms of $\mathbf{z}_1, \mathbf{z}_2, \dot{\theta}_1, \dot{\theta}_2$ only. The total angular momentum H

of the linkage is then given by:

$$H = H_1 + H_2 + H_3 = K_3\dot{\theta}_1 + K_4\dot{\theta}_2 \tag{15}$$

where K_3 and K_4 are written as:

$$\begin{aligned} K_3 &= a_1\mathbf{z}_1 + a_2\mathbf{z}_1^{-1} + a_3\mathbf{z}_1\mathbf{z}_2^{-1} + a_4\mathbf{z}_1^{-1}\mathbf{z}_2 + a_5 \\ K_4 &= b_1\mathbf{z}_2 + b_2\mathbf{z}_2^{-1} + b_3\mathbf{z}_1\mathbf{z}_2^{-1} + b_4\mathbf{z}_1^{-1}\mathbf{z}_2 + b_5 \end{aligned} \tag{16}$$

where constants a_i, b_i can be obtained from (12), (13), (14), (15).

2.5 Dynamic balancing

In our settings, a mechanism is said to be force balanced if the centre of mass of the mechanism remains stationary for infinitely many configurations (i.e. infinitely many choices of the joint angles). From (9), this condition can be formulated as:

$$F = \mathbf{F}_1\mathbf{z}_1 + \mathbf{F}_2\mathbf{z}_2 - \mathbf{rS}' = 0 \tag{17}$$

where $\mathbf{rS}' = \mathbf{rS}M - \mathbf{F}_3$ is a constant. In other words, the expression $\mathbf{F}_1\mathbf{z}_1 + \mathbf{F}_2\mathbf{z}_2$ must be constant. A mechanism is said to be dynamically balanced [15] if the centre of mass remains fixed (force balancing) and the total angular momentum is zero at all times, i.e.,

$$H = K_3\dot{\theta}_1 + K_4\dot{\theta}_2 = 0 \tag{18}$$

Therefore, (3), (4), (17), (18) have to be satisfied. Among these four equations, only two (4) and (18) depend (linearly) on the joint angular velocities and they can be rewritten in the following form:

$$\begin{bmatrix} K_1 & K_2 \\ K_3 & K_4 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{19}$$

If the rank of the matrix $A = \begin{bmatrix} K_1 & K_2 \\ K_3 & K_4 \end{bmatrix}$ is 2 and since the system is homogeneous, then the only solution is $\dot{\theta}_1 = \dot{\theta}_2 = 0$. In other words, the linkage is not moving. Therefore we must have:

$$K := det(A) = K_1 K_4 - K_2 K_3 = 0 \tag{20}$$

We therefore obtain a set of 3 algebraic equations (3),(17),(20) in terms of the unit complex variable $\mathbf{z}_1, \mathbf{z}_2$ and independent from the joint angular velocities. We introduce the quantifier $\underset{\infty}{\exists}$ for "*there exists infinitely many*". Using this notation, the force and moment balancing problems can be formulated as follow:

PROBLEM FORMULATION

Let $(S^1)^2 = \{(\mathbf{z}_1, \mathbf{z}_2) \in \mathbb{C} \mid |\mathbf{z}_1| = 1 \text{ and } |\mathbf{z}_2| = 1\}$. Let G, F and K as defined in equations (3), (17) and (20) respectively.

Force balancing:

Find all possible geometric and static parameters such that:

$$\underset{\infty(\mathbf{z}_1,\mathbf{z}_2)\in(S^1)^2}{\exists} G(\mathbf{z}_1, \mathbf{z}_2) \Rightarrow F(\mathbf{z}_1, \mathbf{z}_2) = 0 \tag{21}$$

Moment balancing:

Find all possible geometric, static and dynamic parameters such that:

$$\underset{\infty(\mathbf{z}_1,\mathbf{z}_2)\in(S^1)^2}{\exists} G(\mathbf{z}_1, \mathbf{z}_2) \Rightarrow F(\mathbf{z}_1, \mathbf{z}_2) = K(\mathbf{z}_1, \mathbf{z}_2) = 0 \tag{22}$$

Using Theorem 1, we can reformulate the balancing problem as a factorisation problem of Laurent polynomials [1] . A proof of this theorem can be found in [10].

Theorem 1 *Let G be an irreducible Laurent polynomial. Let F be a Laurent polynomial(not necessarily irreducible). The following are equivalent:*

(1) $\underset{\infty(\mathbf{z}_1,\mathbf{z}_2)\in(S^1)^2}{\exists} G(\mathbf{z}_1, \mathbf{z}_2) = 0 \Rightarrow F(\mathbf{z}_1, \mathbf{z}_2) = 0$
(2) $\exists$ *Laurent polynomial* $K(\mathbf{z}_1, \mathbf{z}_2)$ *such that* $F = G \cdot K$

[1] Laurent polynomials are polynomials in which the exponents can be negative integers.

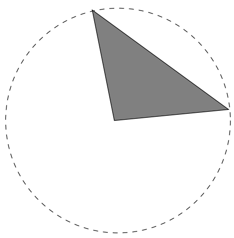


Fig. 5. Degenerated case.

3 Classification of planar four-bar linkages

In this section, a complete characterisation of dynamically balanced planar four-bar linkages is given. We first start with the degenerated case for which one of the body lengths is 0. Then we investigate the case for which the geometric constraint is irreducible. Finally, we derive the conditions for all cases for which the geometric constraint is reducible: the parallelogram, the deltoid and the rhomboid.

3.1 Degenerated case

If one of the l_i (i=1,2,3) is zero and $d \neq 0$, clearly the linkage cannot move since it has a degree of freedom of zero. Therefore, the only case to consider is the case when $d = 0$. If all other lengths are non-zero (i.e. $l_1 \neq 0$, $l_2 \neq 0$, $l_3 \neq 0$), we obtain a triangle rotating about a fixed point (the origin) (see Figure 5). Clearly, this linkage is force balanced if and only if the centre of mass of the linkage is at the origin. The same conditions are obtained if $d = 0$ and one of the l_i is also equal to 0. In this case, the linkage is a pendulum. It is not possible to dynamically balance such degenerated linkages since all parts are rotating in the same direction.

3.2 Irreducible case

Assume G, as given in (3), is an irreducible polynomial and the kinematic parameters are all strictly positive. In this case, G_1, G_2 and G_3 are also different from zero and the coefficients of all monomials of G (except the constant term) are of the form G_iG_j and cannot vanish for any choice of the geometric parameters. Based on Theorem 1 and since G is irreducible, the linkage is force balanced if and only if there exists a polynomial H such that

$$F = GH \tag{23}$$

where F is given by (17). By observation of the degree of polynomials G and F in variables $\mathbf{z}_1$ and $\mathbf{z}_2$, it is clear that there exists no non-zero polynomial H such that (23) is fulfilled. Therefore H must be the zero polynomial and $F = 0$. We obtain the conditions $\mathbf{F}_1 = \mathbf{F}_2 = 0$ which corresponds to the conditions derived by Berkof and Lowen [3]. In this case, these conditions are necessary and sufficient.

For the dynamic balancing, the constraint derived in (20) is more complex. Using a variant of the polynomial division algorithm [10], it is possible to eliminate the variables $\mathbf{z}_1$ and $\mathbf{z}_2$ to obtain a set of constraints in terms of the design parameters. Combining these constraints with the force balancing constraints described above, we obtain a set of equalities and inequalities (due to physical constraints) for the linkage to be dynamically balanced. Among these constraints, we have

$$l_2^2 J_3 + l_3^2 J_2 = 0 \tag{24}$$

where $J_2 = m_2 r_2^2 + I_2$ and $J_3 = m_3 r_3^2 + I_3$. Therefore $I_2 = I_3 = r_2 = r_3 = 0$ which is physically not possible. Therefore, if G is irreducible, a planar four-bar linkage cannot be dynamically balanced.

3.3 Reducible cases

For some specific choices of the geometric parameters, the geometric constraint defined in (3) can be factored in several irreducible components. For example, if $l_1 = l_2 \neq l_3 = d$, the geometric constraint can be factored as:

$$G = \frac{-l_1}{\mathbf{z}_1\mathbf{z}_2} G_A G_B = \frac{-l_1}{\mathbf{z}_1\mathbf{z}_2} \left(d\mathbf{z}_1\mathbf{z}_2 + l_1\mathbf{z}_1 - l_1\mathbf{z}_2 - 2\right)\left(\mathbf{z}_1 - \mathbf{z}_2\right) \qquad (25)$$

The geometric constraint is therefore fulfilled if at least one of the components, either G_A or G_B is zero. Each of these components is called a kinematic mode. The different modes are shown in Table 2. For a detailed description of the possible decompositions, we refer to [6] (page 426) and [10].

For the reducible cases, deriving the balancing conditions is made easier by the fact that all polynomials representing a kinematic mode (Table 2) are linear either in $\mathbf{z}_1$ or $\mathbf{z}_2$ or both. For example, for the parallelogram in mode A, the corresponding polynomial $d\mathbf{z}_1\mathbf{z}_2 + l_1\mathbf{z}_1 - l_1\mathbf{z}_2 - d$ is linear if considered as a polynomial in variable $\mathbf{z}_1$. One can solve for $\mathbf{z}_1$ in terms of $\mathbf{z}_2$ and substitute into the dynamic balancing equation (20) to obtain a univariate polynomial in $\mathbf{z}_2$, which should vanish for infinitely many values of $\mathbf{z}_2$. Therefore all coefficients, which are expressions in terms of the design parameters, of this univariate polynomial must vanish, which gives a set of equations in terms of the design parameters. Below, details are given for the derivation of necessary and sufficient conditions for the dynamic balancing for all reducible cases. The case of the Deltoid-2 is not given explicitely since it is symmetric to the case of the Deltoid-1.

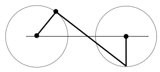
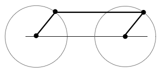
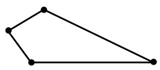
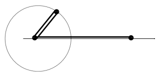
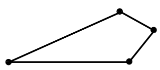
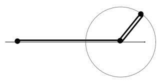
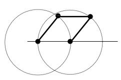
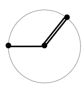
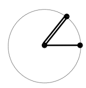
Case	Mode A	Mode B	Mode C
Parallelogram $l_1 = l_2 \neq l_3 = d$	 $d\mathbf{z}_1\mathbf{z}_2 + l_1\mathbf{z}_1 - l_1\mathbf{z}_2 - d = 0$	 $\mathbf{z}_1 - \mathbf{z}_2 = 0$	
Deltoid-1 $l_1 = l_3 \neq l_2 = d$	 $-d\mathbf{z}_1\mathbf{z}_2 - d\mathbf{z}_1 + l_1\mathbf{z}_2 + l_1\mathbf{z}_1^2 = 0$	 $\mathbf{z}_2 + 1 = 0$	
Deltoid-2 $l_1 = d \neq l_2 = l_3$	 $d\mathbf{z}_1\mathbf{z}_2 + l_2\mathbf{z}_1 - d\mathbf{z}_2 - l_2\mathbf{z}_2^2 = 0$	 $\mathbf{z}_1 - 1 = 0$	
Rhomboid $l_1 = l_2 = l_3 = d$	 $\mathbf{z}_1 - \mathbf{z}_2 = 0$	 $\mathbf{z}_1 - 1 = 0$	 $\mathbf{z}_2 + 1 = 0$

Table 2

Kinematic modes.

Parallelogram - Mode A

Here we have $l_3 = d$, $l_1 = l_2 =: l$ with $l \neq d$. Using similar argument as in the irreducible case, the force balancing constraints are given by $\mathbf{F}_1 = \mathbf{F}_2 = 0$. In order to decribe the solutions for the dynamic balancing, we introduce another set of parameters, namely

$$\mathbf{q}_i := m_i r_i \mathbf{p}_i, J_i := I_i + m_i r_i^2, i = 1, 2, 3.$$

Parameters $I_i, r_i, \mathbf{p}_i$ can then be eliminated easily and the balancing conditions become linear in $m_i, \mathbf{q}_i, J_i, i = 1, 2, 3$. For the dynamic balancing, we have the following constraints on the $\mathbf{q}_i$:

$$\begin{aligned} \mathbf{q}_1 &= \frac{l}{d}\mathbf{q}_3 - l m_3 \\ \mathbf{q}_2 &= -\frac{l}{d}\mathbf{q}_3 \end{aligned} \tag{26}$$

and 2 constraints relating the J_i:

$$\begin{aligned} J_1 &= \frac{d^2 + l^2}{d}\mathbf{q}_3 - J_3 - l^2 m_3 \\ J_2 &= \frac{d^2 - l^2}{d}\mathbf{q}_3 - J_3 \end{aligned} \tag{27}$$

A first consequence is that $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ must be real. The parameters fulfill also the inequality constraints

$$m_i > 0, J_i m_i - |\mathbf{q}_i|^2 > 0 \tag{28}$$

for $i = 1, 2, 3$. In particular, J_1 and J_2 must be positive. From (27), we get an upper bound for m_3, which must be larger than the lower bound from (28) ($i = 3$). This yields

$$(d\mathbf{q}_3 - J_3)(dJ_3 - l^2\mathbf{q}_3) > 0. \tag{29}$$

It follows that J_3 is contained in the open interval $(d\mathbf{q}_3, \frac{l^2}{d}\mathbf{q}_3)$. (Note that $\frac{d^2-l^2}{d}\mathbf{q}_3 > 0$ as a consequence of (27), that is why we know which of the two interval boundaries is bigger.) Then $\mathbf{q}_3 > 0$ and $d > l$ follows. From $J_2 > 0$ and (27), we get

$$\frac{l^2}{d}\mathbf{q}_3 < J_3 < \frac{d^2 - l^2}{d}\mathbf{q}_3, \tag{30}$$

from which $d \geq \sqrt{2}l$ follows.

Conversely, if $d \geq \sqrt{2}l$, then we can choose $\mathbf{q}_3 > 0$ arbitrarily and J_3 subject to (30), and m_3 between the upper and lower bound for m_3 derived above. Then (27) determines J_1 and J_2, which will then be positive and (26) determines $\mathbf{q}_1$ and $\mathbf{q}_2$, and finally m_1 and m_2 can be chosen so that inequality (28) is fulfilled.

In Table 3, an example of a dynamically balanced linkage in mode A is shown. For these design parameters, the geometric constraint (3) becomes:

$$G = \frac{-1}{16}\left(4\mathbf{z}_1\mathbf{z}_2 + \mathbf{z}_1 - \mathbf{z}_2 - 4\right)\left(\mathbf{z}_1 - \mathbf{z}_2\right) = 0 \tag{31}$$

and (20) can be written as

$$K = \frac{-3}{512}\frac{m_1\left(4\mathbf{z}_1\mathbf{z}_2 + \mathbf{z}_1 - \mathbf{z}_2 - 4\right)\left(\mathbf{z}_1 + \mathbf{z}_2\right)\left(\mathbf{z}_1^2 + 14\mathbf{z}_1\mathbf{z}_2 + \mathbf{z}_2^2\right)}{\mathbf{z}_1^2\mathbf{z}_2^2} \tag{32}$$

For this mode, the corresponding factor in the geometric constraint equation is $4\mathbf{z}_1\mathbf{z}_2 + \mathbf{z}_1 - \mathbf{z}_2 - 4$ and appears as a factor in K. Therefore, in this kinematic mode, K is always zero. Note that this linkage is also force balanced since $\mathbf{F}_1 = \mathbf{F}_2 = 0$.

Body i	l_i	m_i	r_i	p_i	I_i
$i=1$	1	m_1	$\frac{1}{2}$	-1	$\frac{3m_1}{4}$
$i=2$	1	$\frac{m_1}{3}$	$\frac{1}{2}$	-1	$\frac{5m_1}{4}$
$i=3$	4	$\frac{2m_1}{3}$	1	1	$\frac{m_1}{2}$

Body i	l_i	m_i	r_i	p_i	I_i
$i=1$	1	m_1	$\frac{1}{2}$	-1	$\frac{m_1}{4}$
$i=2$	4	$\frac{m_1}{3}$	2	1	$\frac{m_1}{3}$
$i=3$	1	$\frac{m_1}{3}$	$\frac{1}{2}$	-1	$\frac{3m_1}{4}$

Table 3

Example of dynamically balanced linkages: On the left for the Parallelogram (Mode A), and on the right for the Deltoid-1 (Mode A).

Parallelogram - Mode B

In this mode, $\mathbf{z}_1 = \mathbf{z}_2$. In order for the center of mass to be stationary, we must have:

$$\mathbf{F}_1\mathbf{z}_1 + \mathbf{F}_2\mathbf{z}_2 = \mathbf{F}_1\mathbf{z}_1 + \mathbf{F}_2\mathbf{z}_1 = (\mathbf{F}_1 + \mathbf{F}_2)\mathbf{z}_1 = 0 \tag{33}$$

and therefore $\mathbf{F}_1 + \mathbf{F}_2 = 0$. This is the case that was found in [8]. In this mode, the linkage cannot be dynamically balanced. This can be proven formally using polynomial division and is a direct consequence of the fact that all bodies of the linkage rotate in the same direction.

Deltoid-1 - Mode A

For the Deltoid-1 case, $l_2 = d$ and $l_1 = l_3 =: l$ with $l \neq d$. Using similar argument as in the force balancing of the parallelogram in mode A, it can be easily shown that sufficient and necessary conditions for the force balancing are $\mathbf{F}_1 = \mathbf{F}_2 = 0$.

This is the second case where we get solutions that are physically realizable for the dynamic balancing. Again, we introduce the parameters $\mathbf{q}_i$ and J_i and eliminate $I_i, r_i, \mathbf{p}_i$ for $i = 1, 2, 3$. The balancing conditions are:

$$\mathbf{q}_1 = \mathbf{q}_3 - l m_3, \mathbf{q}_2 = -\frac{d}{l}\mathbf{q}_3, \tag{34}$$

$$J_1 = J_3 - l^2 m_3, J_2 = \frac{l^2 - d^2}{l}\mathbf{q}_3 - J_3. \tag{35}$$

It follows that $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ must be real. The parameters fulfill again the inequality constraints

$$m_i > 0, J_i m_i - |\mathbf{q}_i|^2 > 0 \tag{36}$$

for $i = 1, 2, 3$; again it follows that J_1 and J_2 must be positive. From equation (35), we get an upper bound for m_3, which must be larger than the lower bound given by (36) ($i = 3$). This yields

$$(J_3 - l q_3)(J_3 + l\mathbf{q}_3) > 0. \tag{37}$$

This is equivalent to the statement $J_3 > |l\mathbf{q}_3|$. From (35), we obtain an upper bound for J_3, namely $\left|\frac{d^2-l^2}{l}\mathbf{q}_3\right|$. The lower bound must be larger than the upper bound, hence $\mathbf{q}_3 < 0$ and $d \geq \sqrt{2}l$.

Conversely, if $d \geq \sqrt{2}l$, then we can choose $\mathbf{q}_3 < 0$ arbitrarily and J_3 between $-(l\mathbf{q}_3)$ and $\frac{d^2-l^2}{l}\mathbf{q}_3$. Then we choose m_3 between the upper and lower bound for m_3 derived above. Next, (35) determines J_1 and J_2, which will then be positive. Then (34) determines $\mathbf{q}_1$ and $\mathbf{q}_2$, and finally m_1 and m_2 can be chosen so that inequality (36) is fulfilled.

An example of such a dynamically balanced linkage is shown in Table 3 (right). For these parameters, we have

$$G = -4\,\frac{(\mathbf{z}_2 + 1)\,(\mathbf{z}_2 - 4\,\mathbf{z}_1\,\mathbf{z}_2 + {\mathbf{z}_1}^2 - 4\,\mathbf{z}_1)}{\mathbf{z}_1\,\mathbf{z}_2} \tag{38}$$

$$K = 4\,\frac{\left(\mathbf{z}_2 - 4\,\mathbf{z}_1\,\mathbf{z}_2 + {\mathbf{z}_1}^2 - 4\,\mathbf{z}_1\right)\left(4\,\mathbf{z}_2\,{\mathbf{z}_1}^2 + 4\,{\mathbf{z}_1}^2 - \mathbf{z}_1\,\mathbf{z}_2 + {\mathbf{z}_2}^2\mathbf{z}_1 - 4\,{\mathbf{z}_2}^3 - 4\,{\mathbf{z}_2}^2\right)}{{\mathbf{z}_2}^2{\mathbf{z}_1}^2} \tag{39}$$

Therefore, in Mode A (i.e.: $\mathbf{z}_2 - 4\,\mathbf{z}_1\,\mathbf{z}_2 + {\mathbf{z}_1}^2 - 4\,\mathbf{z}_1 = 0$), we obtain that $K = 0$. Moreover, $\mathbf{F}_1 = \mathbf{F}_2 = 0$ and the linkage is dynamically balanced.

Deltoid-1 - Mode B

In this mode, we have $\mathbf{z}_1 = 1$. Therefore the position of the centre of mass is fixed if the following expression is constant for all $\mathbf{z}_2$:

$$\mathbf{F}_1\mathbf{z}_1 + \mathbf{F}_2\mathbf{z}_2 = \mathbf{F}_1 + \mathbf{F}_2\mathbf{z}_2 \tag{40}$$

Therefore, $\mathbf{F}_2 = 0$. This is the only condition for the force balancing of the deltoid in this kinematic mode. Moreover, dynamic balancing is not possible.

Rhomboid

In the case of the rhomboid, all lengths are equal (i.e. $l_1 = l_2 = l_3 = d$). The three kinematic modes correspond respectively to the parallelogram mode B, the deltoid-1 mode B and the deltoid-2 mode B. Therefore, the balancing constraints can be easily derived from these cases and are summarized in Table 4.

4 Conclusion

The complete characterisation of force and moment balanced planar four-bar linkages was given in this paper and is summarized in Table 4. It was formally

Case	Kinematic mode	Force balancing	Moment balancing
Irreducible		$\mathbf{F}_1 = \mathbf{F}_2 = 0$	no
Parallelogram	A	$\mathbf{F}_1 = \mathbf{F}_2 = 0$	possible iff $d \geq \sqrt{2}\, l_2$
	B	$\mathbf{F}_1 + \mathbf{F}_2 = 0$	no
Deltoid-1	A	$\mathbf{F}_1 = \mathbf{F}_2 = 0$	possible iff $d \geq \sqrt{2}\, l_3$
	B	$\mathbf{F}_1 = 0$	no
Deltoid-2	A	$\mathbf{F}_1 = \mathbf{F}_2 = 0$	possible iff $d \geq \sqrt{2}\, l_3$
	B	$\mathbf{F}_2 = 0$	no
Rhomboid	A	$\mathbf{F}_1 = 0$	no
	B	$\mathbf{F}_2 = 0$	no
	C	$\mathbf{F}_1 + \mathbf{F}_2 = 0$	no

Table 4

Balancing constraints for planar four-bar mechanisms.

proven that this set is complete and that no other balanced four-bar linkages can be found without including additional linkages or counterrotations. It is pointed out that these simple balanced mechanisms can be combined to build more complex planar and spatial balanced mechanisms as shown in [9], thereby leading to potential applications in advanced mechanisms and robotics.

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